Computation of Eigenvalues and Eigenvectors of a Mistuned Bladed Disk Via Unidirectional Taylor Series Expansions

Alok Sinha
Professor of Mechanical Engineering
The Pennsylvania State University,
University Park, PA 16802

This paper deals with the computation of eigenvalues and eigenvectors of a mistuned bladed disk. First, the existence of derivatives of repeated eigenvalues and corresponding eigenvectors is discussed. Next, an algorithm is developed to compute these derivatives. It is shown how a Taylor series expansion can be used to efficiently compute eigenvalues and eigenvectors of a mistuned system. Numerical examples are presented to corroborate the validity of theoretical analysis. [DOI: 10.1115/1.3142863]

1 Introduction

Natural frequencies and mode shapes are computed by solving the following eigenvalue problem:

$$K\mathbf{v} = \lambda M\mathbf{v} \tag{1}$$

where K and M are the symmetric stiffness and mass matrices of the bladed disk, respectively. And, λ and $\mathbf{v}$ are the eigenvalues and eigenvectors of the system, respectively. For a perfectly tuned system, the number of repeated eigenvalue sets equals $(n-1)/2$ and $(n-2)/2$ for odd and even numbers of blades n. This also implies that the number of unrepeated eigenvalue sets equals 1 and 2 for odd and even numbers of blades n. For odd n, the eigenvector corresponding to the unrepeated eigenvalue represents 0 deg interblade phase angle tuned mode. For even n, the eigenvectors corresponding to unrepeated eigenvalues represent 0 deg and 180 deg interblade phase angle tuned modes.

For repeated eigenvalues, eigenvectors are not unique. If $\mathbf{v}_i$ and $\mathbf{v}_j$ are two independent eigenvectors corresponding to a repeated eigenvalue λ_0,

$$K(\alpha\mathbf{v}_i + \beta\mathbf{v}_j) = \lambda_0 M(\alpha\mathbf{v}_i + \beta\mathbf{v}_j) \tag{2}$$

In other words, any linear combination of $\mathbf{v}_i$ and $\mathbf{v}_j$ is also an eigenvector.

Let ΔK and ΔM be the perturbations in mass and stiffness matrices due to mistuning:

$$(K_t + \Delta K)\mathbf{v} = \lambda(M_t + \Delta M)\mathbf{v} \tag{3}$$

where K_t and M_t are the stiffness and mass matrices of the perfectly tuned system, respectively. Because of perturbations in mass and stiffness matrices, repeated eigenvalues for the tuned system split and the mistuned system has distinct eigenvalues and unique eigenvectors. Xiangjun and Shijing [1] commented that the eigenvector corresponding to a repeated eigenvalue is a discontinuous function of system parameters. Applying results in Ref. [2], both eigenvalues and eigenvectors should be analytic with respect to a parameter on which perturbations of mass and stiffness matrices depend. Zhang and Wang [3] developed an analytical approach to compute the derivatives of repeated eigenvalues and corresponding eigenvectors of a nondefective matrix. One of their important contributions is to show that there exists a particular linear combination of eigenvectors $\mathbf{v}_i$ and $\mathbf{v}_j$, which is differentiable. However, with respect to an arbitrary choice of this linear combination, the eigenvector corresponding to a repeated eigenvalue is discontinuous as described by Xiangjun and Shijing [1]. Shapiro [4,5] used a multidimensional Taylor series to compute mistuned eigenvalues. He also showed that the eigenvalue of a mistuned system is a continuous function of mistuned parameters, and it can appear to be discontinuous because of mode switching. However, none of the cited papers [1–5] has dealt with the computation of mistuned eigenvectors via the Taylor series expansion.

For a mistuned bladed disk, one of the main goals is to compute its natural frequencies and mode shapes. Since natural frequencies and mode shapes of a tuned system can be calculated from sector analyses in ANSYS or NASTRAN, it is further desired that the computational algorithm is based on natural frequencies and mode shapes of a tuned system. To accomplish this task, an algorithm has been developed in this paper to compute mistuned natural frequencies and mode shapes on the basis of a Taylor series expansion, which utilizes the derivatives of eigenvalues and eigenvectors. Since there are many parameters that can independently change in a mistuned bladed disk, it is determined if one can use a multidimensional Taylor series. It is shown how an unidirectional Taylor series can be used to efficiently compute natural frequencies and mode shapes of a mistuned system in general. In addition, the validity of the linearization approach is examined.

2 Derivatives of Repeated Eigenvalues and Corresponding Eigenvectors

Let $\mathbf{v}_i$ and $\mathbf{v}_{i+1}$ be the two independent eigenvectors of a perfectly tuned system corresponding to a repeated eigenvalue λ_0. Define

$$X = \begin{bmatrix} \mathbf{v}_i & \mathbf{v}_{i+1} \end{bmatrix} \tag{4}$$

The original choice of eigenvectors may not be differentiable [3]. Therefore, combinations of these two eigenvectors are described as

$$Z = X\Gamma \tag{5}$$

where Γ is a square matrix of dimension 2 with the following property:

$$\Gamma^{-1} = \Gamma^T \tag{6}$$

Now from Eq. (1),

$$KZ = MZ\Lambda \tag{7}$$

where

$$\Lambda = \lambda_0 I_2 \tag{8}$$

2.1 Derivatives of Eigenvalues and Identification of Differentiable Eigenvectors.

Differentiating Eq. (7) with respect to an independent parameter r,

$$(K - \lambda_0 M)\frac{dZ}{dr} + \left(\frac{dK}{dr} - \lambda_0\frac{dM}{dr}\right)Z = MZ\frac{d\Lambda}{dr} \tag{9}$$

where

$$\frac{d\Lambda}{dr} = \begin{bmatrix} \frac{d\lambda_{0,1}}{dr} & 0 \\ 0 & \frac{d\lambda_{0,2}}{dr} \end{bmatrix} \tag{10}$$

Premultiplying Eq. (9) by Z^T,

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$$\frac{d\Lambda}{dr} = Z^T\left(\frac{dK}{dr} - \lambda_0 \frac{dM}{dr}\right)Z \tag{11}$$

Substituting Eq. (5) into Eq. (11), and using Eq. (6),

$$X^T\left(\frac{dK}{dr} - \lambda_0 \frac{dM}{dr}\right)X\Gamma = \Gamma\frac{d\Lambda}{dr} \tag{12}$$

Hence, the matrices Γ and $d\Lambda/dr$ are obtained by solving the eigenvalue/eigenvector problem (12).

2.2 Derivatives of Eigenvectors. Substituting Eq. (11) into Eq. (9),

$$(K - \lambda_0 M)\frac{dZ}{dr} = (MZZ^T - I)\left(\frac{dK}{dr} - \lambda_0 \frac{dM}{dr}\right)Z \tag{13}$$

The dimension of the null space of $(K-\lambda_0 M)$ is 2, and independent vectors in the null space [6] are columns of the matrix Z. Therefore, a general solution of Eq. (13) can be written as

$$\frac{dZ}{dr} = W + ZS \tag{14}$$

where W is a particular solution of Eq. (13), and ZS is the homogeneous solution where the coefficient matrix S is determined from the second derivatives of eigenvalues. Differentiating Eq. (9) with respect to r,

$$(K - \lambda_0 M)\frac{d^2Z}{dr^2} + \left(\frac{d^2K}{dr^2} - \lambda_0 \frac{d^2M}{dr^2}\right)Z + 2\left(\frac{dK}{dr} - \lambda_0 \frac{dM}{dr}\right)\frac{dZ}{dr}$$

$$= MZ\frac{d^2\Lambda}{dr^2} + 2M\frac{dZ}{dr}\frac{d\Lambda}{dr} + 2\frac{dM}{dr}Z\frac{d\Lambda}{dr} \tag{15}$$

Premultiplying Eq. (15) by Z^T,

$$\frac{d^2\Lambda}{dr^2} = Z^T\left(\frac{d^2K}{dr^2} - \lambda_0 \frac{d^2M}{dr^2}\right)Z + 2Z^T\left(\frac{dK}{dr} - \lambda_0 \frac{dM}{dr}\right)\frac{dZ}{dr}$$

$$- 2Z^TM\frac{dZ}{dr}\frac{d\Lambda}{dr} - 2Z^T\frac{dM}{dr}Z\frac{d\Lambda}{dr} \tag{16}$$

Substituting Eq. (14) into Eq. (16) and using Eq. (11),

$$2S\frac{d\Lambda}{dr} - 2\frac{d\Lambda}{dr}S + \frac{d^2\Lambda}{dr^2} = U \tag{17}$$

where

$$U = Z^T\left(\frac{d^2K}{dr^2} - \lambda_0 \frac{d^2M}{dr^2}\right)Z + 2Z^T\left(\frac{dK}{dr} - \lambda_0 \frac{dM}{dr}\right)W - 2Z^TMW\frac{d\Lambda}{dr}$$

$$- 2Z^T\frac{dM}{dr}Z\frac{d\Lambda}{dr} \tag{18}$$

Let s_{ij} and u_{ij} be elements of matrices S and U in ith row and jth column, respectively. Then, equating off-diagonal elements on both sides of Eq. (17),

$$s_{12} = \frac{u_{12}}{2\left(\dfrac{d\lambda_{0,2}}{dr} - \dfrac{d\lambda_{0,1}}{dr}\right)} \tag{19}$$

and

$$s_{21} = \frac{u_{21}}{2\left(\dfrac{d\lambda_{0,1}}{dr} - \dfrac{d\lambda_{0,2}}{dr}\right)} \tag{20}$$

Diagonal elements of the matrix S are obtained from the following normalization condition:

$$\mathbf{z}_i^T M \mathbf{z}_i = 1 \tag{21}$$

where $\mathbf{z}_i$ is the ith column of the matrix Z. Differentiating Eq. (21) with respect to r,

$$2\mathbf{z}_i^T M \frac{d\mathbf{z}_i}{dr} + \mathbf{z}_i^T \frac{dM}{dr}\mathbf{z}_i = 0 \tag{22}$$

From Eq. (14),

$$\frac{d\mathbf{z}_i}{dr} = \mathbf{w}_i + Z\mathbf{s}_i \tag{23}$$

where $\mathbf{w}_i$ and $\mathbf{s}_i$ are ith column of the matrices W and S, respectively. Substituting Eq. (23) into Eq. (22),

$$s_{ii} = -\mathbf{z}_i^T M\mathbf{w}_i - 0.5\mathbf{z}_i^T\frac{dM}{dr}\mathbf{z}_i, \quad i = 1 \text{ and } 2 \tag{24}$$

Having obtained the matrix S, the second derivative of eigenvalues can be obtained from Eq. (16).

3 Taylor Series Expansion

3.1 Multidimensional Taylor Series Expansion of an Eigenvalue. Let ξ_i, $i=1,2,\ldots,\ell$, be ℓ independent random variables describing ΔK and ΔM. Each of these random variables will have n different values, $\xi_{i1},\xi_{i2},\ldots,\xi_{in}$ in the mistuned system. Assuming that there is only one random variable ξ_1,

$$\lambda = \lambda_t + \sum_{j=1}^{n}\frac{\partial\lambda}{\partial\xi_{1j}}\xi_{1j} + \frac{1}{2}\sum_{j=1}^{n}\frac{\partial^2\lambda}{\partial\xi_{1j}^2}\xi_{1j}^2 + \sum_{i=1}^{n}\sum_{j=i+1}^{n}\frac{\partial^2\lambda}{\partial\xi_{1i}\,\partial\xi_{1j}}\xi_{1i}\xi_{1j} + \ldots \tag{25}$$

For a mistuned system, it is quite typical to have $\Sigma_{j=1}^{n}\xi_{1j}=0$. Therefore, mistuned eigenvalues will depend on second-order terms in the multidimensional Taylor series expansion (25). For a repeated eigenvalue, pure second-order derivatives can be obtained from Eq. (16), and there will be a need to develop a similar analytical expression for mixed second-order partial derivatives. It should be noted that second-order derivatives of mass and stiffness matrices are present in Eq. (16) and also in equation for unrepeated eigenvalues [7]. While using ANSYS or NASTRAN, derivatives of mass and stiffness matrices with respect to mistuned parameters may have to be evaluated numerically by finite differences. In this case, numerical efforts can be quite excessive because of a large number of second-order terms. Therefore, a unidirectional Taylor series expansion of an eigenvalue is developed next.

3.2 Unidirectional Taylor Series Expansion of an Eigenvalue and a Discontinuous Eigenvector. For a single random variable case, let the mistuning parameters be

$$\xi_{11},\xi_{12},\ldots,\xi_{1n} \tag{26}$$

Based on the values of these parameters for a mistuned bladed disk, the following vector in the parameter space can be defined as

$$\mathbf{p} = r\boldsymbol{\chi} \tag{27}$$

where

$$\mathbf{p} = [\xi_{11} \quad \xi_{12} \quad \cdots \quad \xi_{1n}]^T \tag{28}$$

$$\boldsymbol{\chi} = [\chi_{11} \quad \chi_{12} \quad \cdots \quad \chi_{1n}]^T \tag{29}$$

$$r = \left[\sum_{i=1}^{n}\xi_{1i}^2\right]^{0.5} \tag{30}$$

Note that χ is a unit vector. Then, the derivatives of mass and stiffness matrices along the vector $\mathbf{p}$ can be calculated as follows:

$$\frac{dM}{dr} = \sum_{i=1}^{n}\frac{\partial M}{\partial\xi_{1i}}\chi_{1i} \tag{31}$$

and

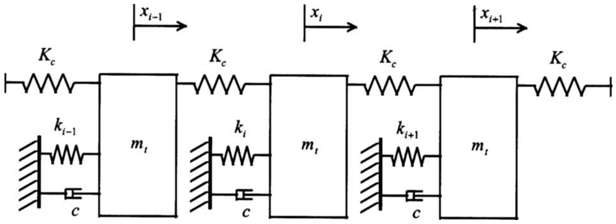

Fig. 1 Model of a bladed disk assembly

$$\frac{dK}{dr} = \sum_{i=1}^{n} \frac{\partial K}{\partial \xi_{1i}} \chi_{1i} \tag{32}$$

These derivatives of mass and stiffness matrices are required for the computation of the derivatives of eigenvalues and eigenvectors. Defining Taylor series expansions of eigenvalues and eigenvectors in terms of a perturbation along the vector $\mathbf{p}$:

$$\lambda = \lambda_t + \frac{d\lambda}{dr} r + \frac{1}{2}\frac{d^2\lambda}{dr^2} r^2 + \ldots \tag{33}$$

and

$$\mathbf{v} = \mathbf{v}_t + \frac{d\mathbf{v}}{dr} r + \frac{1}{2}\frac{d^2\mathbf{v}}{dr^2} r^2 + \ldots \tag{34}$$

Eigenvalues and eigenvectors of the mistuned bladed disk are computed by substituting the value of r, Eq. (30), in Eqs. (33) and (34). It should be noted that λ_t and $\mathbf{v}_t$ are the eigenvalue and the eigenvector of the tuned system, respectively. In the case of a repeated eigenvalue, $\mathbf{v}_t$ is a differential eigenvector obtained from the solution of Eq. (12).

4 Numerical Results

The model shown in Fig. 1 considers only one mode of vibration per blade [7]. Modal mass and stiffness of each blade are represented by m_t and k_i, respectively. The phenomenon of mistuning has been simulated by considering the variations in modal stiffnesses only. The structural coupling between adjacent blades due to the disk flexibility is represented by K_c. Also, $i+1=1$ when $i=n$, and $i-1=n$ when $i=1$. Mistuning parameters are as follows:

$$\xi_{1i} = k_i - k_t, \quad i = 1, 2, \ldots, n \tag{35}$$

The mass m_t and stiffness of the tuned system k_t are 0.0114 kg and 430,000 N/m, respectively. The coupling stiffness K_c $=45{,}430$ N/m. Using Eq. (12), differentiable eigenvectors have been computed for repeated eigenvalues for the unit mistuning vector χ, Table 1. Columns of Table 2 are fourth eigenvectors of a mistuned system for different values of r, which define the mis-

Table 1 Elements of a unit mistuning vector

χ_{11}	χ_{12}	χ_{13}	χ_{14}	χ_{15}	χ_{16}	χ_{17}	χ_{18}	χ_{19}	χ_{110}
−0.1252	−0.3195	0.1885	0.5637	−0.2069	−0.3207	−0.0933	0.2511	−0.3235	−0.4549

Table 2 Fourth eigenvector of a mistuned system ($\bar{r}=r/2.6407$)

	$\bar{r}$									
	-1×10^4	-8×10^3	-6×10^3	-4×10^3	-2×10^3	2×10^3	4×10^3	6×10^3	8×10^3	1×10^4
x_1	4.1299	4.1430	4.1555	4.1671	4.1774	4.1928	4.1973	4.1993	4.1985	4.1949
x_2	1.3420	1.3178	1.2877	1.2512	1.2081	1.1099	1.0365	0.9651	0.8867	0.8017
x_3	−3.0783	−3.1522	−3.2291	−3.3084	−3.3892	−3.5510	−3.6299	−3.7059	−3.7776	−3.8440
x_4	−2.8453	−2.9432	−3.0387	−3.1310	−3.2193	−3.3808	−3.4522	−3.5163	−3.5722	−3.6195
x_5	2.3092	2.1480	1.9790	1.8026	1.6192	1.2330	1.0314	0.8251	0.6149	0.4015
x_6	4.5039	4.4458	4.3847	4.3209	4.2546	4.1157	4.0437	3.9705	3.8963	3.8215
x_7	1.2233	1.1971	1.1772	1.1639	1.1574	1.1658	1.1809	1.2032	1.2325	1.2688
x_8	−3.7061	−3.6716	−3.6307	−3.5834	−3.5299	−3.4054	−3.3352	−3.2605	−3.1816	−3.0992
x_9	−2.8985	−2.9836	−3.0667	−3.1477	−3.2264	−3.3770	−3.4490	−3.5189	−3.5869	−3.6534
x_{10}	1.4280	1.4229	1.4202	1.4203	1.4232	1.4382	1.4506	1.4663	1.4856	1.5084

Table 3 Fourth eigenvector of a mistuned system via linearization ($\bar{r}=r/2.6407$)

	$\bar{r}$									
	-1×10^4	-8×10^3	-6×10^3	-4×10^3	-2×10^3	2×10^3	4×10^3	6×10^3	8×10^3	1×10^4
x_1	4.1474	4.1551	4.1629	4.1706	4.1784	4.1939	4.2016	4.2094	4.2171	4.2248
x_2	1.4260	1.3724	1.3188	1.2652	1.2116	1.1044	1.0508	0.9972	0.9436	0.8900
x_3	−3.0648	−3.1460	−3.2271	−3.3082	−3.3893	−3.5515	−3.6326	−3.7138	−3.7949	−3.8760
x_4	−2.8985	−2.9794	−3.0602	−3.1411	−3.2220	−3.3837	−3.4646	−3.5455	−3.6264	−3.7072
x_5	2.3951	2.2019	2.0087	1.8155	1.6223	1.2359	1.0427	0.8495	0.6563	0.4631
x_6	4.5336	4.4641	4.3946	4.3251	4.2556	4.1166	4.0471	3.9776	3.9081	3.8387
x_7	1.1370	1.1412	1.1454	1.1496	1.1538	1.1622	1.1664	1.1706	1.1749	1.1791
x_8	−3.7819	−3.7196	−3.6573	−3.5950	−3.5327	−3.4081	−3.3458	−3.2835	−3.2212	−3.1589
x_9	−2.9264	−3.0017	−3.0770	−3.1523	−3.2276	−3.3782	−3.4535	−3.5287	−3.6040	−3.6793
x_{10}	1.3918	1.3993	1.4067	1.4142	1.4217	1.4366	1.4441	1.4515	1.4590	1.4665

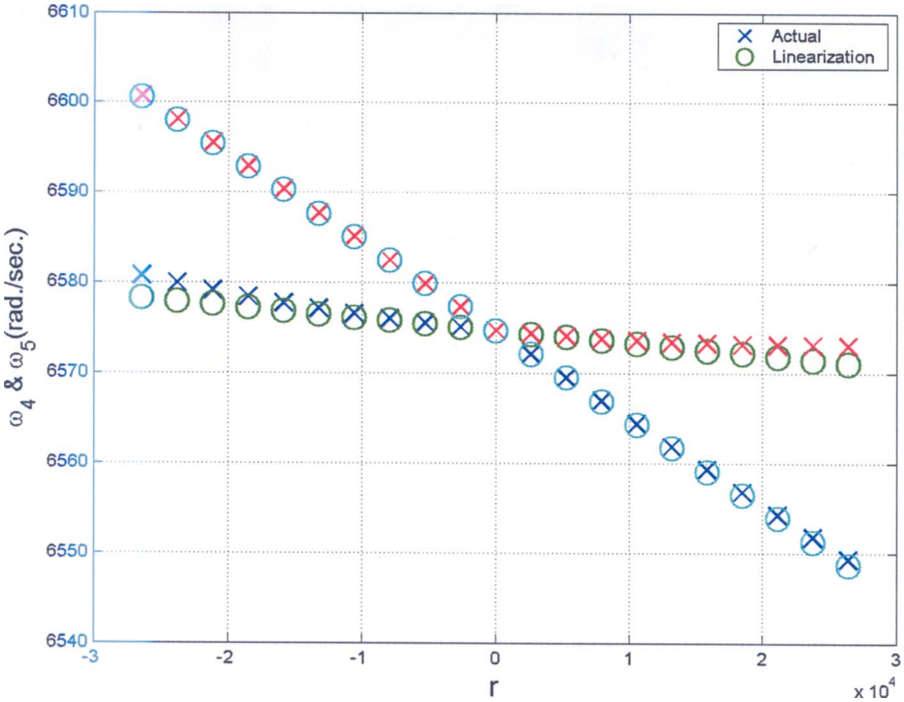

Fig. 2 Prediction of ω_4 and ω_5 via linearization (range of r axis: -3×10^4 to $+3\times10^4$)

tuning vector **p**, Eq. (27). Table 3 contains fourth eigenvectors predicted by linearization, i.e., after neglecting second and higher derivatives in Eq. (34). The derivative of the eigenvector has been computed from Eq. (23). Comparing columns in Tables 2 and 3, the linearized analysis is found to yield fairly accurate results.

Natural frequencies ω_4 and ω_5 are plotted as functions of r in Fig. 2. Once again, the linearized analysis yields fairly good prediction of natural frequencies. The first derivative in Eq. (33) has been obtained via Eq. (12).

5 Conclusions

All eigenvectors corresponding to a repeated eigenvalue, in general, are discontinuous functions of mistuning parameters with respect to a constant interblade phase angle tuned mode. However, differentiable eigenvectors can be uniquely obtained, and unidirectional Taylor series expansions can be used to predict variations in repeated eigenvalues and corresponding eigenvectors in a computationally efficient manner. For typical values of mistuning parameters of a simple model of bladed disk (Fig. 1), the linearization method based on this unidirectional Taylor series is found to be accurate.

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